The University of Western Australia
DEPARTMENT OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2022 Problems

The Competition begins Friday, 1 July and ends Friday, 16 September 2022.

You may use any source of information except other people. In particular, you may use computer packages (though contributed solutions are expected to have complete mathematical proofs). Extensions and generalisations of any problem are invited and are taken into account when assessing solutions. Also, elegance of solution is particularly favoured and desired. We reserve the right to not read a solution if it is "messy"; we expect neat solutions (so perhaps avoid submitting a first draft).

Solutions are to be emailed as PDF attachments to Greg Gamble (`greg.gamble@uwa.edu.au`) **by 4 pm on Friday, 16 September.**

Due to COVID, alas, there is no money for prizes. But we do it for glory, right?

Instructions for solutions: Include the following information in the body of your email:

name,
student ID number,
home address (optional),
e-mail address,
university where enrolled,
number of years you have been attending any tertiary institution, and
list of the questions completed and attached as PDFs to your email.

Please scan each question to a separate PDF file and name the file according to the protocol: $\langle$`YourLastName`$\rangle\langle n\rangle$`.pdf` where $\langle n\rangle$ is the number of the question.
Before you scan, please ensure you have started each new problem on a new page, and that at the top of each page is the number of the page, and your full name.

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Easy but complex

Suppose $z_1, z_2 \in \mathbb{C}$ such that

$$|z_1| = |z_1 + z_2| = 3$$
$$|z_1 - z_2| = 3\sqrt{3}.$$

Determine $\lfloor \log_3 |(z_1\overline{z_2})^{2022} + (\overline{z_1}z_2)^{2022}| \rfloor$, where $\lfloor x \rfloor$ is the *floor* of x (the largest integer less than or equal to $x \in \mathbb{R}$), and $|z|$ and $\overline{z}$ are, respectively, the *modulus* and *conjugate* of $z \in \mathbb{C}$.

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2. Annually cubic

Let $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha^3 - 3\alpha^2 + 23\alpha = 20, \text{ and}$$
$$\beta^3 - 3\beta^2 + 23\beta = 22.$$

Find $\alpha + \beta$.

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3. Parabolic sum

For each $n \in \mathbb{N}$, the parabola

$$y = (n^2 + n)x^2 + (2n + 1)x + 1,$$

cuts the x-axis at α_n and β_n.

Find $\sum_{n=1}^{2022} \alpha_n \beta_n$.

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4. Alternating sums

Let the decimal digit representation of $N \in \mathbb{N}$ be $\overline{a_n a_{n-1} \ldots a_1 a_0}$, i.e.

$$N = a_0 + 10a_1 + 10^2 a_2 + \cdots + 10^n a_n.$$

and let $A(N)$ be the *alternating sum of digits* of N. i.e.

$$A(N) = a_0 - a_1 + a_2 - \cdots + (-1)^n a_n.$$

Also, we define $A(0) = 0$, and for $N < 0$, $A(N) = -A(-N)$ (what one gets by thinking of all the digits of N as negative).

Suppose $\alpha = 2022^{2022}$, $A(\alpha) = \beta$, and $A(\beta) = \gamma$.

Knowing that $A(\gamma)$ is negative, what is $A(\gamma)$?

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5. Perimetric area

The diameter of the circumcircle of acute triangle ABC is 4, and its area is S, with X, Y, Z points on sides BC, CA, AB respectively.

Prove that AX, BY, CZ are altitudes of $\triangle ABC$ if and only if $S = ZY + YX + XZ$.
