The University of Western Australia
DEPARTMENT OF MATHEMATICS AND STATISTICS

BLAKERS MATHEMATICS COMPETITION

2020 Problems

The Competition begins Friday, 22 May and ends Friday, 7 August 2020.

You may use any source of information except other people. In particular, you may use computer packages (though contributed solutions are expected to have complete mathematical proofs). Extensions and generalisations of any problem are invited and are taken into account when assessing solutions. Also, elegance of solution is particularly favoured and desired. We reserve the right to not read a solution if it is "messy"; we expect neat solutions (so perhaps avoid submitting a first draft).

Solutions are to be emailed as PDF attachments to Greg Gamble (`greg.gamble@uwa.edu.au`) **by 4 pm on Friday, 7 August.**

Remember, you don't have to solve all the problems to win prizes!

Instructions for solutions: Include the following information in the body of your email:

name,
student ID number,
home address (optional),
e-mail address,
university where enrolled,
number of years you have been attending any tertiary institution, and
list of the questions completed and attached as PDFs to your email.

Please scan each question to a separate PDF file and name the file according to the protocol: ⟨`YourLastName`⟩⟨n⟩`.pdf` where ⟨n⟩ is the number of the question.
Before you scan, please ensure you have started each new problem on a new page, and that at the top of each page is the number of the page, and your full name.

Note. Our convention is that $\mathbb{N} = \{1, 2, \dots\}$ (the positive integers).

1. Equilaterally so

Let x be the side length of an equilateral triangle ABC, and suppose points P and Q lie in the interior of ABC such that $PQ = 1$, $AP = AQ = \sqrt{7}$ and $BP = CQ = 2$, where line segments BP and CQ do not intersect.

What is the value of x?

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2. Empowered reciprocals

Let x, y, z be non-zero real numbers such that $x + y + z \neq 0$. If

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x + y + z},$$

for which integers $n > 1$, can we deduce that

$$\frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} = \frac{1}{(x + y + z)^n}?$$

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3. Really covered

The set of all points of the plane $\mathbb{R}^2$ whose coordinates are both rational is denoted by $\mathbb{Q}^2$. Does the set of all intersection points of line segments that join two points of $\mathbb{Q}^2$, cover the plane $\mathbb{R}^2$?

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4. Coin toss salad

Alice and Bob toss a fair coin many times. If a head appears, Alice gives Bob \$1, and if a tail appears, Bob gives Alice \$1. Initially, Alice has \$$a$, and Bob has \$$b$.

If the game continues until one of the two players has lost everything, what is the probability that Alice wins?

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5. Sum functions making a difference

Determine all nonconstant, infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) - f(x - y) = f'(x)f'(y),$$

for all $x, y \in \mathbb{R}$.

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6. Volume of discussion

Given two real numbers $a, b \in [0, 1]$ such that $a + b = 1$, let D be a closed disc of radius a in $\mathbb{R}^3$ and let D^* be the set of points of $\mathbb{R}^3$ whose distance to D is at most b.

Note. The distance of a point P to D is by definition the minimum of the distances from P to points in D.

For what values of a and b is the maximum volume of D^* attained?

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7. Mind your ps and qs

Determine the pairs (p, q) of positive integers such that

$$1^p + 2^p + \cdots + n^p = (1 + 2 + \cdots + n)^q,$$

for all $n \in \mathbb{N}$.

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8. A story well-told

Correspondents $C_1, C_2, \ldots, C_n$ communicate with each other by letters. Each correspondent knows 1 detail of a certain story that has piqued their interest, but the n details of their collective knowledge are all different. Whenever one of the correspondents sends a letter to another correspondent, they tell the other correspondent everything they know of the story, at the time of writing the letter.

What is the minimum number of letters that the n correspondents should send to one another so that each of them gets to know all the details of the story?

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9. Row distinction

For which values of the integer n is there an $n \times n$ square matrix, whose entries are either 0 or 1, such that the sums of the entries in the n rows are all different and the sums of the entries in the n columns are all equal?

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10. Circular intimacy

In the plane, let α, β, γ be three circles of equal radii, that touch each other pairwise (externally), and are located inside a fourth circle K that touches each of them. From an arbitrary point P on K, tangents are drawn to the circles α, β, γ, to meet these circles at points A, B, C respectively.

For which such points P is one of the distances PA, PB, PC equal to the sum of the other two?
